Geometric distance-regular graphs without 4-claws

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Abstract

A non-complete distance-regular graph Γ is called geometric if there exists a set $\mathcal{C}$ of Delsarte cliques such that each edge of Γ lies in a unique clique in $\mathcal{C}$. In this paper, we determine the non-complete distance-regular graphs satisfying $\max\{3, \frac{8}{3}(a_1+1)\} < k < 4a_1 + 10 - 6c_2$. To prove this result, we first show by considering non-existence of 4-claws that any non-complete distance-regular graph satisfying $\max\{3, \frac{8}{3}(a_1+1)\} < k < 4a_1+10-6c_2$ is a geometric distance-regular graph with smallest eigenvalue -3. Moreover, we classify the geometric distance-regular graphs with smallest eigenvalue -3. As an application, 7 feasible intersection arrays in the list of [7, Chapter 14] are ruled out.

1 Introduction

Let Γ be a distance-regular graph with valency k and let $\theta_{\min} = \theta_{\min}(\Gamma)$ be its smallest eigenvalue. Any clique C in Γ satisfies

$$|C| \leq 1 - \frac{k}{\theta_{\min}} \tag{1}$$

(see [7, Proposition 4.4.6 (i)]). This bound (1) is due to Delsarte, and a clique C in Γ is called a *Delsarte clique* if C contains exactly $1 - \frac{k}{\theta_{\min}}$ vertices. Godsil [11] introduced the following notion of a geometric distance-regular graph. A non-complete distance-regular graph Γ is called *geometric* if there exists a set $\mathcal{C}$ of Delsarte cliques such that each edge of Γ lies in a unique Delsarte clique in $\mathcal{C}$. In this case, we say that Γ is geometric with respect to $\mathcal{C}$.

There are many examples of geometric distance-regular graphs such as bipartite distance-regular graphs, the Hamming graphs, the Johnson graphs, the Grassmann graphs and regular near $2D$-gons.

In particular, the local structure of geometric distance-regular graphs play an important role in the study of spectral characterization of some distance-regular graphs. In [1], we show that for given

integer $D \geq 2$, any graph cospectral with the Hamming graph $H(D,q)$ is locally the disjoint union of D copies of the complete graph of size $q-1$, for q large enough. By using this result and [4], we show in [1] that the Hamming graph $H(3,q)$ with $q \geq 36$ is uniquely determined by its spectrum.

Neumaier [17] showed that except for a finite number of graphs, any geometric strongly regular graph with a given smallest eigenvalue $-m$, $m > 1$ integral, is either a Latin square graph or a Steiner graph (see [17] and Remark 4.4 for the definitions).
An *n-claw* is an induced subgraph on $n+1$ vertices which consists of one vertex of valency n and n vertices of valency 1. Each distance-regular graph without 2-claws is a complete graph. Note that for any geometric distance-regular graph Γ with respect to $\mathcal{C}$ a set of Delsarte cliques, the number of Delsarte cliques in $\mathcal{C}$ containing a fixed vertex is $-\theta_{\min}(\Gamma)$. Hence any geometric distance-regular graph with smallest eigenvalue -2 contains no 3-claws. Blokhuis and Brouwer [6] determined the distance-regular graphs without 3-claws.
Yamazaki [20] considered distance-regular graphs which are locally a disjoint union of three cliques of size a_1+1, and these graphs for $a_1 \geq 1$ are geometric distance-regular graphs with smallest eigenvalue -3.

In Theorem 4.3, we determine the geometric distance-regular graphs with smallest eigenvalue -3. We now state our main result of this paper.

Theorem 1.1 *Let Γ be a non-complete distance-regular graph. If Γ satisfies*

$$\max\{3, \frac{8}{3}(a_1+1)\} < k < 4a_1 + 10 - 6c_2$$

then Γ is one of the following.

(i) A Steiner graph $S_3(\alpha-3)$, i.e., a geometric strongly regular graph with parameters $\left(\frac{(2\alpha-3)(\alpha-2)}{3}, 3\alpha-9, \alpha, 9\right)$, where $\alpha \geq 36$ and $\alpha \equiv 0, 2 \pmod 3$.

(ii) A Latin square graph $LS_3(\alpha)$, i.e., a geometric strongly regular graph with parameters $(\alpha^2, 3(\alpha-1), \alpha, 6)$, where $\alpha \geq 24$.

(iii) The generalized hexagon of order $(8,2)$ with $\iota(\Gamma) = \{24, 16, 16; 1, 1, 3\}$.

(iv) One of the two generalized hexagons of order $(2,2)$ with $\iota(\Gamma) = \{6, 4, 4; 1, 1, 3\}$.

(v) A generalized octagon of order $(4,2)$ with $\iota(\Gamma) = \{12, 8, 8, 8; 1, 1, 1, 3\}$.

(vi) The Johnson graph $J(\alpha, 3)$, where $\alpha \geq 20$.

(vii) $D = 3$ and $\iota(\Gamma) = \{3\alpha+3, 2\alpha+2, \alpha+2-\beta; 1, 2, 3\beta\}$, where $\alpha \geq 6$ and $\alpha \geq \beta \geq 1$.

(viii) The halved Foster graph with $\iota(\Gamma) = \{6, 4, 2, 1; 1, 1, 4, 6\}$.

(ix) $D = \mathtt{h} + 2 \geq 4$ and

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \text{for } 1 \leq i \leq \mathtt{h} \\ (2, 2\alpha+\beta-1, \alpha-\beta+2) & \text{for } i = \mathtt{h}+1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \text{for } i = \mathtt{h}+2 \end{cases}, \text{ where } \alpha \geq \beta \geq 2.$$

(x) $D = \mathtt{h}+2 \geq 3$ *and*

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \textit{for } 1 \leq i \leq \mathtt{h} \\ (1, \alpha+2\beta-2, 2\alpha-2\beta+4) & \textit{for } i = \mathtt{h}+1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \textit{for } i = \mathtt{h}+2 \end{cases}, \textit{ where } \alpha \geq \beta \geq 2.$$

*(xi) A distance-*2 *graph of a distance-biregular graph with vertices of valency* 3 *and*

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \textit{for } 1 \leq i \leq \mathtt{h} \\ (1, \alpha+2, 2\alpha) & \textit{for } i = \mathtt{h}+1 \\ (4, 2\alpha-1, \alpha) & \textit{for } \mathtt{h}+2 \leq i \leq D-2 \\ (4, 2\alpha+\beta-3, \alpha-\beta+2) & \textit{for } i = D-1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \textit{for } i = D \end{cases}, \textit{ where } \alpha \geq \beta \textit{ and } \beta \in \{2, 3\}.$$

Examples of non-complete distance-regular graphs with valency $k > \max\{3, \frac{8}{3}(a_1+1)\}$ include Johnson graphs $J(n,e)$ $\Big((n \geq 20$ and $e = 3)$, $(n \geq 11$ and $e = 4)$ or $(n \geq 2e$ and $e \geq 5)\Big)$, Hamming graphs $H(d,q)$ $\Big((d = 3$ and $q \geq 3)$ or $(d \geq 4$ and $q \geq 2)\Big)$ and Grassmann graphs $\begin{bmatrix} V \\ e \end{bmatrix}$ $\Big((e = 2$ and $q \geq 4)$ or $(e \geq 3$ and $q \geq 2)\Big)$, where $n \geq 2e$ and V is an n-dimensional vector space over $\mathbb{F}_q$ the finite field of $q(\geq 2)$ elements (see [7, Chapter 9] for more information on these examples). Except $J(n,3)$ $(n \geq 20)$ and $H(3,q)$ $(q \geq 3)$, all the above examples contain 4-claws. Whereas, $J(n,3)$ $(n \geq 20)$ and $H(3,q)$ $(q \geq 3)$ are geometric distance-regular graphs with smallest eigenvalue -3.

In Section 3, we prove Theorem 3.1 which gives a sufficient condition, $\max\{3, \frac{8}{3}(a_1+1)\} < k < 4a_1 + 10 - 6c_2$, for geometric distance-regular graphs with smallest eigenvalue -3. We first show in Theorem 3.2 that for any distance-regular graph satisfying $k > \max\{3, \frac{8}{3}(a_1+1)\}$, the statement that Γ has no 4-claws is equivalent to the statement that Γ is geometric with smallest eigenvalue -3. By using Theorem 3.2, we will prove Theorem 3.1. As an application of Theorem 3.2, we can show non-existence of a family of distance-regular graphs with feasible intersection arrays. For example, in the list of [7, Chapter 14], the 7 feasible intersection arrays in Theorem 3.5 are ruled out.

In Section 4, we determine the geometric distance-regular graphs with smallest eigenvalue -3 in Theorem 4.3. By using Theorem 3.1 and Theorem 4.3, we will prove Theorem 1.1.

2 Preliminaries

All graphs considered in this paper are finite, undirected and simple (for unexplained terminology and more details, see [7]).
For a connected graph Γ, distance $d_\Gamma(x,y)$ between any two vertices x, y in the vertex set $V(\Gamma)$ of Γ is the length of a shortest path between x and y in Γ, and denote by $D(\Gamma)$ the diameter of Γ (i.e., the maximum distance between any two vertices of Γ). For any vertex $x \in V(\Gamma)$, let $\Gamma_i(x)$ be the

set of vertices in Γ at distance precisely i from x, where i is a non-negative integer not exceeding $D(\Gamma)$. In addition, define $\Gamma_{-1}(x) = \Gamma_{D(\Gamma)+1}(x) := \emptyset$ and $\Gamma_0(x) := \{x\}$. For any distinct vertices $x_1, x_2, \ldots, x_j \in V(\Gamma)$, define

$$\Gamma_1(x_1, \ldots, x_j) := \Gamma_1(x_1) \cap \Gamma_1(x_2) \cap \cdots \cap \Gamma_1(x_j).$$

A *clique* is a set of pairwise adjacent vertices. A graph Γ is called *locally* G if any local graph of Γ (i.e., the local graph of a vertex x is the induced subgraph on $\Gamma_1(x)$) is isomorphic to G, where G is a graph. The *adjacency matrix* $A(\Gamma)$ of a graph Γ is the $|V(\Gamma)| \times |V(\Gamma)|$-matrix with rows and columns are indexed by $V(\Gamma)$, and the (x, y)-entry of $A(\Gamma)$ equals 1 whenever $d_\Gamma(x, y) = 1$ and 0 otherwise. The eigenvalues of Γ are the eigenvalues of $A(\Gamma)$.
A connected graph Γ is called a *distance-regular graph* if there exist integers $b_i(\Gamma)$, $c_i(\Gamma)$, $i = 0, 1, \ldots, D(\Gamma)$, such that for any two vertices x, y at distance $i = d_\Gamma(x, y)$, there are precisely $c_i(\Gamma)$ neighbors of y in $\Gamma_{i-1}(x)$ and $b_i(\Gamma)$ neighbors of y in $\Gamma_{i+1}(x)$. In particular, Γ is regular with valency $k(\Gamma) := b_0(\Gamma)$. The numbers $c_i(\Gamma), b_i(\Gamma)$ and $a_i(\Gamma) := k(\Gamma) - b_i(\Gamma) - c_i(\Gamma)$ $(0 \leq i \leq D(\Gamma))$ (i.e., the number of neighbors of y in $\Gamma_i(x)$ for $d_\Gamma(x, y) = i$) are called the *intersection numbers* of Γ. Note that $b_{D(\Gamma)}(\Gamma) = c_0(\Gamma) = a_0(\Gamma) := 0$ and $c_1(\Gamma) = 1$. In addition, we define $k_i(\Gamma) := |\Gamma_i(x)|$ for any vertex x and $i = 0, 1, \ldots, D(\Gamma)$. The array $\iota(\Gamma) = \{b_0(\Gamma), b_1(\Gamma), \ldots, b_{D(\Gamma)-1}(\Gamma); c_1(\Gamma), c_2(\Gamma), \ldots, c_{D(\Gamma)}(\Gamma)\}$ is called the *intersection array* of Γ. In addition, we define the number

$$\mathtt{h}(\Gamma) := |\{j \mid (c_j, a_j, b_j) = (c_1, a_1, b_1),\ 1 \leq j \leq D(\Gamma) - 1\}| \tag{2}$$

which is called the *head* of Γ.
A regular graph Γ on v vertices with valency $k(\Gamma)$ is called a *strongly regular graph* with parameters $(v, k(\Gamma), \lambda(\Gamma), \mu(\Gamma))$ if there are two constants $\lambda(\Gamma) \geq 0$ and $\mu(\Gamma) > 0$ such that for any two distinct vertices x and y, $|\Gamma_1(x, y)|$ equals $\lambda(\Gamma)$ if $d_\Gamma(x, y) = 1$ and $\mu(\Gamma)$ otherwise.
When there are no confusion, we omit $\sim_\Gamma$ and $\sim(\Gamma)$ in each notation for Γ, such as $d_\Gamma(\ ,\ )$, $D(\Gamma)$, $A(\Gamma)$, $\mathtt{h}(\Gamma)$, $k(\Gamma), c_i(\Gamma), b_i(\Gamma), a_i(\Gamma)$, $k_i(\Gamma)$, $\lambda(\Gamma)$ and $\mu(\Gamma)$.
Suppose that Γ is a distance-regular graph with valency $k \geq 2$ and diameter $D \geq 2$. It is well-known that Γ has exactly $D + 1$ distinct eigenvalues which are the eigenvalues of the following tridiagonal matrix

$$L_1(\Gamma) := \begin{pmatrix} 0 & b_0 & & & & & & \\ c_1 & a_1 & b_1 & & & & & \\ & c_2 & a_2 & b_2 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & c_i & a_i & b_i & & \\ & & & & \cdot & \cdot & \cdot & \\ & & & & & c_{D-1} & a_{D-1} & b_{D-1} \\ & & & & & & c_D & a_D \end{pmatrix} \tag{3}$$

(cf. [7, p.128]). In particular, we denote by $\theta_{\min} = \theta_{\min}(\Gamma)$ the smallest eigenvalue of Γ.

3 Distance-regular graphs without 4-claws

In this section, we prove the following theorem which gives a sufficient condition for geometric distance-regular graphs with smallest eigenvalue -3.

Theorem 3.1 *Let Γ be a non-complete distance-regular graph. If Γ satisfies*

$$\max\{3, \frac{8}{3}(a_1+1)\} < k < 4a_1 + 10 - 6c_2 \tag{4}$$

then Γ is a geometric distance-regular graph with smallest eigenvalue -3.

We first show in Theorem 3.2 that for any distance-regular graph satisfying $k > \max\{3, \frac{8}{3}(a_1 + 1)\}$, the statement that Γ has no 4-claws is equivalent to the statement that Γ is geometric with smallest eigenvalue -3. By using Theorem 3.2, we will prove Theorem 3.1. As an application, by considering a restriction on c_2 in Lemma 3.4, we can rule out a family of feasible intersection arrays. In particular, we prove that there are no distance-regular graphs with the intersection arrays in Theorem 3.5.

Theorem 3.2 *Let Γ be a distance-regular graph satisfying $k > \max\{3, \frac{8}{3}(a_1 + 1)\}$. Then the following are equivalent.*
(i) Γ has no 4-claws.
(ii) Γ is a geometric distance-regular graph with smallest eigenvalue -3.

Proof: Let Γ be a distance-regular graph satisfying $k > \max\{3, \frac{8}{3}(a_1 + 1)\}$. Let $\theta_{\min} = \theta_{\min}(\Gamma)$.
(ii)$\Rightarrow$(i): Suppose that Γ is geometric with respect to $\mathcal{C}$ a set of Delsarte cliques and $\theta_{\min} = -3$. Since the number of Delsarte cliques in $\mathcal{C}$ containing a given vertex is $-\theta_{\min}$, the statement (i) follows immediately.
(i)$\Rightarrow$(ii): Suppose that Γ has no 4-claws. Define a *line* to be a maximal clique C in Γ such that C has at least $k - 2(a_1 + 1) + 1$ vertices. Note here that $a_1 \geq 1$ follows, otherwise Γ has a 4-claw from $k > \max\{3, \frac{8}{3}(a_1 + 1)\}$. Hence, $|C| \geq 3$ for any line C in Γ. If there exists a line C satisfying $|C| = 3$, then $a_1 = 1$ and $k = 6$ both hold by $3 \geq k - 2(a_1 + 1) + 1$ and $k > \frac{8}{3}(a_1 + 1)$. By [12, Theorem 1.1], the graph Γ is one of the following.
(a) The generalized quadrangle of order $(2, 2)$.
(b) One of the two generalized hexagons of order $(2, 2)$.
(c) The Hamming graph $H(3, 3)$.
(d) The halved Foster graph.
All the graphs in (a)-(d) are geometric with smallest eigenvalue -3.
In the rest of the proof, we assume that each line contains more than 3 vertices. First, we prove the following claim.

Claim 3.3 *Every edge of Γ lies in a unique line.*

Proof of Claim 3.3: Let (x, y_1) be an arbitrary edge in Γ. As $k \geq 2(a_1 + 1) + 1$, there exists a 3-claw containing x and y_1, say $\{x, y_1, y_2, y_3\}$ induces a 3-claw, where $y_i \in \Gamma_1(x)$ $(i = 1, 2, 3)$. Put $Y_i := \{y_i\} \cup \Gamma_1(x, y_i)$ $(i = 1, 2, 3)$. If there exists a vertex z in $\Gamma_1(x) \setminus \cup_{i=1}^{3} Y_i$, then $\{x, z, y_1, y_2, y_3\}$ induces a 4-claw which is impossible, and therefore $\Gamma_1(x) = \cup_{i=1}^{3} Y_i$ follows. If there exist non-adjacent two vertices v, w in $Y_1 \setminus (Y_2 \cup Y_3)$, then the set $\{x, y_2, y_3, v, w\}$ induces a 4-claw which is a contradiction. Hence $\{x\} \cup (Y_1 \setminus (Y_2 \cup Y_3))$ induces a clique containing the edge (x, y_1), and it satisfies

$$|\{x\} \cup (Y_1 \setminus (Y_2 \cup Y_3))| = |\{x\}| + |\Gamma_1(x)| - |Y_2 \cup Y_3| \geq 1 + k - 2(a_1 + 1).$$

Thus every edge lies in a line.
Assume that there exist two lines C_z and C_w containing the edge (x, y_1), where $z \in C_z$ and $w \in C_w$ are two non-adjacent vertices. Then $a_1 = |\Gamma_1(x, y_1)| \geq 2(k - 2(a_1 + 1) - 1) - (|C_z \cap C_w| - 2)$ implies

$$|C_z \cap C_w| \geq 2k - 5a_1 - 4. \tag{5}$$

In addition, by (5),

$$\begin{aligned} |\Gamma_1(x) \setminus (\Gamma_1(x,z) \cup \Gamma_1(x,w) \cup \{z,w\})| &\geq k - (|\Gamma_1(x,z)| + |\Gamma_1(x,w)| + |\{z,w\}| - (|C_z \cap C_w| - 1)) \\ &\geq k - (2(a_1+1) - (2k - 5a_1 - 5)) \\ &= 3k - 7a_1 - 7. \end{aligned} \tag{6}$$

Since Γ has no 4-claws, $(\{x\} \cup \Gamma_1(x)) \setminus (\Gamma_1(x,z) \cup \Gamma_1(x,w) \cup \{z,w\})$ induces a clique of size at least $3k - 7a_1 - 6$ by (6). Since any clique in Γ has size at most $a_1 + 2$, we have $k \leq \frac{8}{3}(a_1 + 1)$ which is impossible. Hence, the edge (x, y_1) lies in a unique line. Now, Claim 3.3 is proved. ∎

For each vertex $x \in V(\Gamma)$, we define M_x to be the number of lines containing x. Then for any vertex x, we have $M_x \geq 3$ as $k > \frac{8}{3}(a_1 + 1) > 2(a_1 + 1)$, and hence

$$M_x = 3 \text{ for each vertex } x \in V(\Gamma) \tag{7}$$

as $k \geq M_x(k - 2(a_1 + 1))$ holds by Claim 3.3. Let B be the vertex-line incidence matrix (i.e., the $(0,1)$-matrix with rows and columns are indexed by the vertex set and the set of lines of Γ respectively, where (x, C)-entry of B is 1 if the vertex x is contained in the line C and 0 otherwise). By Claim 3.3 and (7), $BB^T = A + 3I$ holds, where B^T is the transpose of B, $A = A(\Gamma)$ and I is the $|V(\Gamma)| \times |V(\Gamma)|$ identity matrix. Since each line contains more than 3 vertices, it follows by double-counting the number of ones in B that the number of lines is strictly less than the number of vertices in Γ. Hence, the matrix BB^T is singular so that 0 is an eigenvalue of BB^T and thus -3 is an eigenvalue of A. As BB^T is positive semidefinite, we find $\theta_{\min} = -3$. Hence it follows by (1), Claim 3.3, (7) and $\theta_{\min} = -3$ that every line has exactly $1 + \frac{k}{3}$ vertices. This proves that Γ is geometric with $\theta_{\min} = -3$. ∎

In [16, Lemma 2], Koolen and Park have shown the following lemma.

Lemma 3.4 *Let Γ be a distance-regular graph with a 4-claw. Then Γ satisfies*

$$c_2 \geq \frac{4a_1 + 10 - k}{6}.$$

Proof: Suppose that $\{x, y_i \mid 1 \leq i \leq 4\}$ induces a 4-claw in Γ, where $y_i \in \Gamma_1(x)$ $(i = 1, 2, 3, 4)$. It follows by the principle of inclusion and exclusion that

$$\begin{aligned} k &\geq |\{y_i \mid 1 \leq i \leq 4\}| + \left|\cup_{i=1}^4 \Gamma_1(x, y_i)\right| \\ &\geq |\{y_i \mid 1 \leq i \leq 4\}| + \sum_{i=1}^4 |\Gamma_1(x, y_i)| - \sum_{1 \leq i < j \leq 4} |\Gamma_1(x, y_i, y_j)| \\ &\geq 4 + 4a_1 - \binom{4}{2}(c_2 - 1), \end{aligned}$$

from which Lemma 3.4 follows. ∎

We now prove our main result of Section 3, Theorem 3.1.

Proof of Theorem 3.1: Suppose that Γ is a non-complete distance-regular graph satisfying (4). Then there are no 4-claws in Γ by Lemma 3.4, so that Γ is geometric with $\theta_{\min}(\Gamma) = -3$ by Theorem 3.2. This completes the proof. ∎

Theorem 3.5 *There are no distance-regular graphs with the following intersection arrays*
(i) $\{55, 36, 11; 1, 4, 45\}$,
(ii) $\{56, 36, 9; 1, 3, 48\}$,
(iii) $\{65, 44, 11; 1, 4, 55\}$,
(iv) $\{81, 56, 24, 1; 1, 3, 56, 81\}$,
(v) $\{117, 80, 32, 1; 1, 4, 80, 117\}$,
(vi) $\{117, 80, 30, 1; 1, 6, 80, 117\}$,
(vii) $\{189, 128, 45, 1; 1, 9, 128, 189\}$.

Proof: Assume that Γ is a distance-regular graph such that its intersection array is one of the 7 intersection arrays (i)-(vii). Since Γ satisfies $k > \frac{8}{3}(a_1 + 1)$, $a_1 \neq 0$ and $\theta_{\min}(\Gamma) \neq -3$, Γ has a 4-claw by Theorem 3.2. It follows by Lemma 3.4 that $c_2 \geq \frac{4a_1+10-k}{6}$ which is impossible. This shows Theorem 3.5. ∎

Remark 3.6 (a) *Koolen and Park [16] showed the non-existence of distance-regular graphs with the intersection array (iii) in Theorem 3.5 and so did Jurišić and Koolen [14] for the intersection arrays (iv)-(vii).*

(b) *Suppose that* Γ *is a distance-regular graph with an intersection array (i), (ii) or (iii) in Theorem 3.5. By [7, Proposition 4.2.17],* Γ_3 *(the graph with the vertices are* $V(\Gamma)$ *and the edges are the* 2*-subsets of vertices at distance* 3 *in* Γ*) is a strongly regular graph with parameters* $(672, 121, 20, 22)$, $(855, 126, 21, 18)$ *or* $(924, 143, 22, 22)$, *respectively. No strongly regular graphs with these parameters are known.*

4 Geometric distance-regular graphs with smallest eigenvalue -3

In this section, we prove Theorem 4.3 in which we determine the geometric distance-regular graphs with smallest eigenvalue -3.

Let Γ be a distance-regular graph with diameter $D = D(\Gamma)$. For any non-empty subset X of $V(\Gamma)$ and for each $i = 0, 1, \ldots, D$, we put

$$X_i := \{x \in V(\Gamma) \mid d(x, X) = i\},$$

where $d(x, X) = \min\{d(x, y) \mid y \in X\}$. Suppose that $C \subseteq V(\Gamma)$ is a Delsarte clique in Γ. For each $i = 0, 1, \ldots, D-1$ and for a vertex $x \in C_i$, define

$$\psi_i(x, C) := |\{z \in C \mid d(x, z) = i\}|\,.$$

The number $\psi_i(x, C)$ $(i = 0, 1, \ldots, D-1)$ depends not on the pair (x, C) but depends only on the distance $i = d(x, C)$ (cf. [2, Section 4] and [10, Section 11.7]). Hence denote

$$\psi_i := \psi_i(x, C) \;\; (i = 0, 1, \ldots, D-1).$$

Now, let Γ be geometric with respect to $\mathcal{C}$ a set of Delsarte cliques. For $x, y \in V(\Gamma)$ with $d(x, y) = i$ $(i = 1, 2, \ldots, D)$, define $\tau_i(x, y; \mathcal{C})$ as the number of cliques C in $\mathcal{C}$ satisfying $x \in C$ and $d(y, C) = i-1$. By [2, Lemma 4.1], the number $\tau_i(x, y; \mathcal{C})$ $(i = 1, 2, \ldots, D)$ depends not on the pair (x, y) and $\mathcal{C}$, but depends only on the distance $i = d(x, y)$. Thus we may put

$$\tau_i := \tau_i(x, y; \mathcal{C}) \; (i = 1, 2, \ldots, D)\,.$$

Note that for any geometric distance-regular graph Γ,

$$\tau_D = -\theta_{\min} \tag{8}$$

holds, where $D = D(\Gamma)$ and $\theta_{\min} = \theta_{\min}(\Gamma)$.
The next lemma is a direct consequence of [2, Proposition 4.2 (i)].

Lemma 4.1 *Let Γ be a geometric distance-regular graph. Then the following hold.*
(i) $b_i = -(\theta_{\min} + \tau_i)\left(1 - \frac{k}{\theta_{\min}} - \psi_i\right)$ $(1 \le i \le D-1)$.
(ii) $c_i = \tau_i \psi_{i-1}$ $(1 \le i \le D)$.

Note that by (8) and Lemma 4.1 (ii), any geometric distance-regular graph with diameter D satisfies

$$c_D = (-\theta_{\min})\psi_{D-1} \ge -\theta_{\min}. \tag{9}$$

Lemma 4.2 *Let Γ be a geometric distance-regular graph. Then*

$$\psi_1 \le \tau_2 \le -\theta_{\min}. \tag{10}$$

In particular, $\psi_1^2 \le c_2 \le \theta_{\min}^2$ holds.

Proof: Let x be a vertex and let C be a Delsarte clique satisfying $x \notin C$. If there are two neighbors y and z of x in C, then two edges (x, y) and (x, z) lie in different Delsarte cliques as Γ is geometric. This shows $\psi_1 \le \tau_2$. Note that the number of Delsarte cliques containing any fixed vertex is $-\theta_{\min}$, so that $\tau_i \le -\theta_{\min}$ for all $i = 1, \ldots, D$. Hence, we find $\psi_1 \le \tau_2 \le -\theta_{\min}$. In particular, it follows by Lemma 4.1 (ii) and (10) that $\psi_1^2 \le \tau_2\psi_1 = c_2 \le \theta_{\min}^2$ holds. ∎

Theorem 4.3 *Let Γ be a geometric distance-regular graph with smallest eigenvalue -3. Then Γ satisfies one of the following.*

(i) *$k = 3$ and Γ is one of the following graphs: the Heawood graph, the Pappus graph, Tutte's 8-cage, the Desargues graph, Tutte's 12-cage, the Foster graph, $K_{3,3}$, $H(3,2)$.*

(ii) *A Steiner graph $S_3(\alpha-3)$, i.e., a geometric strongly regular graph with parameters $\left(\frac{(2\alpha-3)(\alpha-2)}{3}, 3\alpha-9, \alpha, 9\right)$, where $\alpha \geq 6$ and $\alpha \equiv 0, 2 \pmod 3$.*

(iii) *A Latin square graph $LS_3(\alpha)$, i.e., a geometric strongly regular graph with parameters $(\alpha^2, 3(\alpha-1), \alpha, 6)$, where $\alpha \geq 4$.*

(iv) *The generalized $2D$-gon of order $(s,2)$, where $(D,s) = (2,2), (2,4), (3,8)$.*

(v) *One of the two generalized hexagons of order $(2,2)$ with $\iota(\Gamma) = \{6,4,4;1,1,3\}$.*

(vi) *A generalized octagon of order $(4,2)$ with $\iota(\Gamma) = \{12,8,8,8;1,1,1,3\}$.*

(vii) *The Johnson graph $J(\alpha,3)$, where $\alpha \geq 6$.*

(viii) *$D = 3$ and $\iota(\Gamma) = \{3\alpha+3, 2\alpha+2, \alpha+2-\beta; 1, 2, 3\beta\}$, where $\alpha \geq \beta \geq 1$.*

(ix) *The halved Foster graph with $\iota(\Gamma) = \{6,4,2,1;1,1,4,6\}$.*

(x) *$D = \mathtt{h}+2 \geq 4$ and*

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \text{for } 1 \leq i \leq \mathtt{h} \\ (2, 2\alpha+\beta-1, \alpha-\beta+2) & \text{for } i = \mathtt{h}+1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \text{for } i = \mathtt{h}+2 \end{cases}, \text{ where } \alpha \geq \beta \geq 2.$$

(xi) *$D = \mathtt{h}+2 \geq 3$ and*

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \text{for } 1 \leq i \leq \mathtt{h} \\ (1, \alpha+2\beta-2, 2\alpha-2\beta+4) & \text{for } i = \mathtt{h}+1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \text{for } i = \mathtt{h}+2 \end{cases}, \text{ where } \alpha \geq \beta \geq 2.$$

(xii) *A distance-2 graph of a distance-biregular graph with vertices of valency 3 and*

$$(c_i, a_i, b_i) = \begin{cases} (1, \alpha, 2\alpha+2) & \text{for } 1 \leq i \leq \mathtt{h} \\ (1, \alpha+2, 2\alpha) & \text{for } i = \mathtt{h}+1 \\ (4, 2\alpha-1, \alpha) & \text{for } \mathtt{h}+2 \leq i \leq D-2 \\ (4, 2\alpha+\beta-3, \alpha-\beta+2) & \text{for } i = D-1 \\ (3\beta, 3\alpha-3\beta+3, 0) & \text{for } i = D \end{cases}, \text{ where } \alpha \geq \beta \text{ and } \beta \in \{2,3\}.$$

Proof: Let Γ be geometric with respect to $\mathcal{C}$. As $\theta_{\min} = -3$, we have $k \equiv 0 \pmod 3$. If $k = 3$ then Γ satisfies (i) by [5] (cf.[7, Theorem 7.5.1]). In the rest of the proof, we assume $k \geq 6$ and let $D = D(\Gamma)$. We divide the proof into two cases, (**Case 1:** $c_2 \geq 2$) and (**Case 2:** $c_2 = 1$).

Case 1: $c_2 \geq 2$
By (10) with $\theta_{\min} = -3$, we find $\psi_1 \in \{1, 2, 3\}$.

First suppose $\psi_1 = 1$, so that Γ is locally a disjoint union of three cliques of size $a_1 + 1$ and $k = 3(a_1 + 1)$. By [20, Theorem 3.1], Γ satisfies either ($c_2 = 2$ and $2 \leq D \leq 3$) or ($c_2 = 3$ and $D = 2$). If $c_2 = 2$ and $D = 2$ then -3 is not the smallest eigenvalue of the matrix $L_1(\Gamma)$ in (3), which contradicts to $\theta_{\min} = -3$. If $c_2 = 2$ and $D = 3$ then $\tau_2 = 2$ and $\tau_3 = 3$ by Lemma 4.1 (ii) and (8), respectively, and thus $(c_1, a_1, b_1) = (1, a_1, 2a_1 + 2)$, $(c_2, a_2, b_2) = (2, 2a_1 - 1 + \psi_2, a_1 + 2 - \psi_2)$ and $(c_3, a_3, 0) = (3\psi_2, 3a_1 + 3 - 3\psi_2, 0)$ all hold by Lemma 4.1. Now, Γ satisfies (viii). If $c_2 = 3$ and $D = 2$, then Γ is the generalized quadrangle of order $(s, 2)$, where $s = 2, 4$ (cf. [7, Theorem 6.5.1] and [13, Theorem 1]).

Next suppose $\psi_1 = 2$, so that $\tau_2 \in \{2, 3\}$, $b_1 = \frac{2(k-3)}{3}$ and $c_2 = 2\tau_2$ all follow by (10) and Lemma 4.1. If $D \geq 3$ then Γ is the Johnson graph $J(\alpha, 3)$ ($\alpha \geq 6$) of diameter 3 by [15, Theorem 7.1] and [3, Remark 2 (ii)]. Now, we consider $D = 2$. Then, $\tau_2 = 3$ by (8), and Γ is a strongly regular graph with parameters $(a_1^2, 3(a_1 - 1), a_1, 6)$, where $a_1 \geq 4$ as $k \geq 6$ and Γ is geometric. Hence, (iii) follows as Γ is the line graph of a $2-(3\alpha, 3, 1)$-transversal design, where $\mathcal{C}$ and $V(\Gamma)$ are the set of points and lines respectively (See Remark 4.4 (b)).

Finally, we consider $\psi_1 = 3$. Then $c_2 = \tau_2\psi_1 = 9$ holds by Lemma 4.2. From Lemma 4.1 (i) with $\theta_{\min} + \tau_2 = 0$, $D = 2$ follows, and thus $(c_1, a_1, b_1) = (1, a_1, 2a_1 - 10)$ and $(c_2, a_2, b_2) = (9, 3a_1 - 18, 0)$. Since Γ is geometric, Γ is a Steiner graph $S_3(\alpha - 3)$ and Γ satisfies (ii), where the restriction on a_1 is obtained from $k \geq 6$ and the fact that $|V(\Gamma)|$ is a positive integer (See [17, p.396] and Remark 4.4). This completes the proof of **Case 1**.

Case 2: $c_2 = 1$
From the conditions $c_2 = \tau_2\psi_1 = 1$ and $\theta_{\min} = -3$, Γ is locally a disjoint union of three cliques of size $a_1 + 1$. If $a_1 \leq 1$ then $k \in \{3, 6\}$ follows from $|C| \in \{2, 3\}$ for any Delsarte clique C in Γ. By [12], Γ satisfies (v) or (ix).
From now on, we assume $a_1 \geq 2$. First suppose $c_{\mathtt{h}+1} \geq 2$, where $\mathtt{h} = \mathtt{h}(\Gamma)$ is the head of Γ in (2). Then by (9) and [20, Theorem 3.1], Γ satisfies either ($c_{\mathtt{h}+1} = 3$ and $D = \mathtt{h} + 1$) or ($c_{\mathtt{h}+1} = 2$ and $D = \mathtt{h} + 2$). For the case $c_{\mathtt{h}+1} = 3$, Γ is a generalized $2D$-gon of order $(s, 2)$, where $(D, s) = (3, 8), (4, 4)$ (cf. [7, Section 6.5] and [13, Theorem 1]). If $c_{\mathtt{h}+1} = 2$, then we find $\psi_{\mathtt{h}} = 1$ and $\tau_{\mathtt{h}+1} = 2$ by $c_{\mathtt{h}} = \psi_{\mathtt{h}-1}\tau_{\mathtt{h}} = 1$ and

$$a_1 = a_{\mathtt{h}} = \tau_{\mathtt{h}}(a_1 + 1 - \psi_{\mathtt{h}-1}) + (3 - \tau_{\mathtt{h}})(\psi_{\mathtt{h}} - 1),$$

from which (x) holds by (8), Lemma 4.1 and [13, Proposition 2]. Next suppose $c_{\mathtt{h}+1} = 1$. By (9) and [20, Theorem 4.1], Γ satisfies either $D = \mathtt{h} + 2$ or (xii). For the case $D = \mathtt{h} + 2$ with $c_{\mathtt{h}+1} = 1$, (xi) follows by (8) and Lemma 4.1. This completes the proof of Theorem 4.3. ∎

We remark on the distance-regular graphs in Theorem 4.3.

Remark 4.4 *(a) The line graph of a Steiner triple system on* $2\alpha-3$ *points for any integer* $\alpha \geq 6$ *satisfying* $\alpha \equiv 0, 2 \pmod 3$*, which is called a* Steiner graph $S_3(\alpha-3)$*, is a strongly regular graph given in (ii). With the fact that a Steiner triple system on* v *points exists for each integer* v *satisfying* $v \equiv 1$ *or* $3 \pmod 6$*, Wilson showed in [18] and [19] that there are super-exponentially many Steiner triple systems for an admissible number of points, hence so are strongly regular graphs in (ii) (cf. [8, p. 209], [17, Lemma 4.1]).*

(b) The line graph of a $2-(mn, m, 1)$*-transversal design* $(n \geq m+1)$ *is called a Latin square graph* $LS_m(n)$ *(See [17, p.396]). In particular, a Latin square graph* $LS_3(\alpha)$ *is a geometric strongly regular graph in (iii). Since there are more than exponentially many Latin squares of order* α*, so are such strongly regular graphs in (iii) (cf. [8, p. 210], [17, Lemma 4.2]).*

(c) In the list of [7, Chapter 14], only the Hamming graph $H(3, \alpha+2)$*, the Doob graph of diameter* 3 *and the intersection array* $\{45, 30, 7; 1, 2, 27\}$ *satisfy (viii). No distance-regular graph with the last array,* $\{45, 30, 7; 1, 2, 27\}$*, is known. We can also check that if* Γ *satisfies (viii) then the eigenvalues of* Γ *are integers.*

Proof of Theorem 1.1: It is straightforward from Theorem 3.1 and Theorem 4.3. ∎

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References

[1] S. Bang, E. R. van Dam and J. H. Koolen, Spectral characterizaion of the Hamming graphs, *Linear Algebra Appl.* **429(11-12)** (2008), 2678–2686.

[2] S. Bang, A. Hiraki and J. H. Koolen, Delsarte clique graphs, *European J. Combin.* **28(2)** (2007), 501–516.

[3] S. Bang, A. Hiraki and J. H. Koolen, Delsarte set graphs with small c_2, *Graphs Combin.* **26(2)** (2010), 147–162.

[4] S. Bang and J. H. Koolen, Graphs cospectral with $H(3, q)$ which are disjoint unions of at most three complete graphs, *Asian-Eur. J. Math.* **1(2)** (2008), 147–156.

[5] N. L. Biggs, A. G. Boshier and J. Shawe-Taylor, Cubic distance-regular graphs, *J. Lond. Math. Soc. (2)* **33(3)** (1986), 385–394.

[6] A. Blokhuis and A. E. Brouwer, Determination of the distance-regular graphs without 3-claws, *Discrete Math.* **163(1-3)** (1997), 225–227.

[7] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-regular graphs, *Springer-Verlag, Berlin,* 1989.

[8] P. J. Cameron, Strongly regular graphs, in: Topics in algebraic graph theory, L.W. Beineke and R.J. Wilson eds., Encyclopedia of Mathematics and Its Applications **102**, Cambridge University Press, Cambridge, 2004, 203–221.

[9] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Rep. Suppl.* **10** 1973.

[10] C. D. Godsil, Algebraic combinatorics, Champman and Hall Mathematics Series, *Champman and Hall, New York,* 1993.

[11] C. D. Godsil, Geometric distance-regular covers, *New Zealand J. Math.* **22(2)** (1993), 31–38.

[12] A. Hiraki, K. Nomura, H. Suzuki, Distance-regular graphs of valency 6 and $a_1 = 1$, *J. Algebraic Combin.* **11(2)** (2000), 101–134.

[13] A. Hiraki and J. Koolen, The regular near polygons of order $(s, 2)$, *J. Algebraic Combin.* **20(2)** (2004), 219–235.

[14] A. Jurišić and J. H. Koolen, Nonexistence of some antipodal distance-regular graphs of diameter four, *European J. Combin.* **21(8)** (2000), 1039–1046.

[15] J. H. Koolen and S. Bang, On distance-regular graphs with smallest eigenvalue at least $-m$, *J. Combin. Theory Ser. B* **100(6)** (2010), 573–584.

[16] J. H. Koolen and J. Park, Shilla distance-regular graphs, *European J. Combin.* **31(8)** (2010), 2064–2073.

[17] A. Neumaier, Strongly regular graphs with smallest eigenvalue $-m$, *Arch. Math. (Basel)* **33(4)** (1979/80), 392–400.

[18] R. M. Wilson, Nonisomorphic Steiner triple systems, *Math. Z.* **135** (1973/74), 303–313.

[19] R. M. Wilson, An existence theory for pairwise balanced designs. III. Proof of the existence conjectures, *J. Combin. Theory Ser. A* **18** (1975), 71–79.

[20] N. Yamazaki, Distance-regular graphs with $\Gamma(x) \simeq 3 * K_{a+1}$, *European J. Combin.* **16(5)** (1995), 525–536.